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Adomian decomposition Method and Differential Transform Method to solve the Heat Equations with a power nonlinearity

$$u_t(x,t) = u_{xx} + u^m \ , \quad m > 1$$

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Abstract
In this paper, we consider the AdomianDecomposition Method (ADM) and the Differential Transform Method (DTM) for finding approximate and exact solution of the heat equation with a power nonlinearity. Moreover, the reliability and performance of ADM and DTM. Numerical results show that these methods are powerful tools for solving heat equation with a power nonlinearity.

I. Introduction

In this paper, we consider the heat equation with a power nonlinearity

$$u_t(x,t) = u_{xx} + u^m \quad (1.1)$$

subject to the initial condition

$$u(x,0) = f(x) \quad (1.2)$$

where $m > 1$ and the indices t and x denote derivatives with respect to these variables.

Serdal [1], Sujit [2] and Hooman [3] discussed the ADM. This method has been applied to a wide problems in many mathematical and physical areas. In section 2 we using AMD to solve equation (1.1).

Turning the other side of each of the Malek[4], Keskin[5],Saravanan[6] and Mahmoud [8] studied DTM. It is a numerical method based on a Taylor expansion. DTM is important method to solve problems in many important applications. So, in section 3 we discuses how to solve equation (1.1) by DTM.

II. Adomian Decomposition Method

In this section, we explain the main algorithm of ADM for nonlinear heat equations with initial condition. Through the following references [1], [2] and [3] the ADM are studied to find approximate solutions to the equation (1.1).

Equation (1.1) is approximated by an operator in the following form

$$L_t u(x,t) = L_{xx} u + u^m \quad (2.1)$$

Where $L_t = \frac{\partial}{\partial t}$, $L_{xx} = \frac{\partial^2}{\partial x^2}$, and Nu represent the general nonlinear operator u^m . Taking the inverse operator of the operator L_t exists and it defined as

$$L_t^{-1}(\,.\,) = \int_0^t (\,.\,)dt \quad (2.2)$$

Thus, applying the inverse operator L_t^{-1} to equation (2.1) yields

$$L_t^{-1} L_t u(x,t) = L_t^{-1} L_{xx} u + \varepsilon L_t^{-1} u^m \quad (2.3)$$

$$u(x,t) = u(x,0) + L_t^{-1} L_{xx} u + \varepsilon L_t^{-1} u^m. \qquad (2.4)$$

In ADM we represent a solution suppose that

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad (2.5)$$

is a required solution of equation (1.1). A nonlinear term occurs equation (1.1), we can decompose it by using Adomian polynomial, which is given by the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty} \lambda^n u_n\right)\right]_{\lambda=0} \qquad ,n \geq 0 \qquad (2.6)$$

Therefore,

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n \qquad (2.7)$$

where A_n are Adomian polynomials of $u_0\,,\ u_1\,,\ \ldots\ldots, u_n \quad , n \geq 0$ which are calculated by using equation (2.6). we obtain the first few Adomian polynomials for as , $Nu(x,t) = u^m$ as

$$A_0 = u_0^m$$
$$A_1 = m u_0^{m-1}$$
$$A_2 = \frac{m}{2}[(m-1)u_0^{m-2}u_1^2 + 2u_2 u_0^{m-1}]$$
$$\vdots$$

Substituting (2.5) and (2.7) into equation (2.4) we get

$$u(x,t) = u(x,0) + L_t^{-1} L_{xx} \sum_{n=0}^{\infty} u_n(x,t) + L_t^{-1} \sum_{n=0}^{\infty} A_n \ . \qquad (2.8)$$

From equation (2.8) the Adomian decomposition scheme is defined by the recurrent relation

$$u_0(x,0) = f(x)$$

and $\quad u_{n+1}(x,t) = L_t^{-1} L_{xx} u_n(x,t) + \varepsilon L_t^{-1} A_n$ for $n = 0,1,2, \ldots\ldots$

form which

$$u_1(x,t) = L_t^{-1} L_{xx} u_0 + L_t^{-1} A_0$$
$$u_2(x,t) = L_t^{-1} L_{xx} u_1 + L_t^{-1} A_1$$
$$u_3(x,t) = L_t^{-1} L_{xx} u_2 + L_t^{-1} A_2$$
$$\vdots$$
$$u_n(x,t) = L_t^{-1} L_{xx} u_{n-1} + L_t^{-1} A_{n-1}$$

We can estimate the approximate solution ϕ_γ by using the γ- term approximation . That is ,

$$\phi_\gamma = \sum_{n=0}^{\gamma-1} u_n(x,t) \qquad (2.9)$$

Where the components produce as

$$\phi_1 = u_0$$
$$\phi_2 = u_0 + u_1$$
$$\phi_3 = u_0 + u_1 + u_2$$
$$\vdots$$
$$\phi_\gamma = u_0 + u_1 + u_2 + \cdots + u_{\gamma-1}$$

As it is clear from equation (2.5) and (2.9)

$$u(x,t) = \lim_{\gamma \to \infty} \phi_\gamma(x,t)$$

So the decomposition method has been proved to be reliable in handling nonlinear heat equations with initial condition given by equations (1) and (2). Themethod avoids the cumbersome work needed by traditional classical techniques. Furthermore, the method can be extended to solve nonlinear heat equations with a power nonlinearity.

III. Differential Transform Method

In this section and through the references [4], [5] and [6] we introduce the basic definition and the operation of the differential transformation.

If the function $u(x,t)$is analytic and differential continuously with respect to time t and space x in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0} \quad , \qquad (3.1)$$

where the t- dimensional spectrum function $U_k(x)$ is the transformed function.

So the differential inverse transform of $U_k(x)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k \qquad (3.2)$$

From eq. (3.1) and (3.2) the function $u(x,t)$ can be described as

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0} t^k \qquad (3.3)$$

From the above, it can be found that the concept of differential transform method is derived from the power series expansion of a function. The fundamental mathematical operations performed by differential transform method are listed in table 1 below:

Table 1. Differential transform

Functional Form	Transformed Form
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k \pm V_k$
$w(x,t) = \alpha u(x,t)$	$W_k(x) = \alpha U_k \quad ,\alpha\ is\ a\ constant$
$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k-n) \quad ,\delta(k-n) = \begin{cases} 1 & ,k=n \\ 0 & ,k \neq n \end{cases}$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U_{k-n}(x)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} U_r V_{k-r}(x) = \sum_{r=0}^{k} V_r U_{k-r}(x)$
$w(x,t) = \frac{\partial}{\partial x}u(x,t)$	$W_k(x) = \frac{\partial}{\partial x}U_k(x)$
$w(x,t) = \frac{\partial}{\partial x^r}^{r} u(x,t)$	$W_k(x) = (k+1)\ldots(k+r)U_{k+r}(x) = \frac{(k+r)!}{k!}U_{k+r}(x)$
$w(x,t) = [u(x,t)]^m \quad ,m = 1,2,\ldots$	$W_k(x) = \begin{cases} [U_0(x)]^m, & k=0 \\ \sum_{n=1}^{k} \frac{(m+1)n-k}{kU_0(x)} W_{k-n}(x), & k \geq 1 \end{cases}$

From the table 1, we use the diferential transform method to obtain the solution of equation (1.1) and (1.2). By taking the diferential transform on both sides of (1.1) and (1.2) we have

$$(k+1)U_{k+1}(x) = \frac{\partial^2}{\partial x^2}U_k(x) + \varepsilon F_k(x) \tag{3.4}$$

$$U_0(x) = f(x)\,, \tag{3.5}$$

where $F_k(x)$ are

$$F_0(x) = U_0^m(x)$$

$$F_1(x) = mU_0^{m-1}(x)U_1(x)$$

$$F_2(x) = \frac{1}{2}m(m-1)U_0^{m-2}(x)U_1^2(x) + mU_0^{m-1}(x)U_2(x)$$

$$F_3(x) = \frac{1}{6}m(m-1)(m-2)U_0^{m-3}(x)U_1^3(x) + m(m-1)U_0^{m-2}(x)U_2(x)U_1(x) + mU_0^{m-1}(x)U_3(x)$$

$$\vdots$$

From (3.2), we have,

$$u(x,t) = U_0(x) + U_1(x)t + U_2(x)t^2 + U_3(x)t^3 + \cdots + U_n(x)t^n + \cdots \tag{3.6}$$

$$u_n(x,t) = \sum_{k=0}^{n} U_k(x)t^k\,, \tag{3.7}$$

where $n$is order of approximation solution.
Therefore, the exact solution of the equation (1.1) is given by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{3.8}$$

The DTM is successful in solving heat equation with a power nonlinearity. Thus DTM is an important method and reliable and promising method with existing methods.

IV. Applications

Consider the nonlinear heat equation and subject to initial condition

$$u_t(x,t) = u_{xx} - 2u^3 \quad (4.1)$$

$$u(x,0) = \frac{1+2x}{x^2+x+1} \quad (4.2)$$

Where for the exact solution of (4.1) as

$$u(x,t) = \frac{1+2x}{x^2+x+6t+1} \qquad (4.3)$$

Case 1. (By ADM)

In an operator form, (4.1) becomes,

$$u(x,t) = u(x,0) + L_t^{-1}L_{xx}u - 2L_t^{-1}u^3 \quad (2.4)$$

In this case the Adomian Polynomials are

$$A_0 = -2u_0^3$$

$$A_1 = -6u_0^2u_1$$

$$A_2 = -6(u_0u_1^2 + u_0^2u_2)$$

And so on. Therefore, we obtain

$$u_0 = \frac{1+2x}{x^2+x+1}$$

$$u_1 = L_t^{-1}L_{xx}(u_0) - 2L_t^{-1}(u_0^3) = \frac{-6(1+2x)}{(x^2+x+1)^2}t$$

$$u_2 = L_t^{-1}L_{xx}(u_1) - 6L_t^{-1}(u_0^2u_1) = \frac{36(1+2x)}{(x^2+x+1)^3}t^2$$

$$u_3 = L_t^{-1}L_{xx}(u_2) - 6L_t^{-1}(u_0^2u_2 + u_1^2u_0) = \frac{-216(1+2x)}{(x^2+x+1)^4}t^3$$

$$\vdots$$

Substituting into (2.5), we obtain

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$

$$u(x,t) = \frac{1+2x}{x^2+x+1} - \frac{6(1+2x)}{(x^2+x+1)^2}t + \frac{36(1+2x)}{(x^2+x+1)^3}t^2 - \frac{-216(1+2x)}{(x^2+x+1)^4}t^3 + \cdots$$

Case 2. (By DTM)

From the table 1, we use the diferential transform method to obtain the solution of equation (4.1) and (4.2). By taking the diferential transform on both sides of (4.1) and (4.2) we have

$$(k+1)U_{k+1}(x) = \frac{\partial^2}{\partial x^2}U_k(x) - 2F_k(x)$$

$$U_0(x) = \frac{1+2x}{x^2+x+1}$$

where $F_k(x)$ are

$$F_0(x) = U_0^3(x) = \left[\frac{1+2x}{x^2+x+1}\right]^3$$

$$F_1(x) = 3U_0^2(x)U_1(x) = 3\left[\frac{1+2x}{x^2+x+1}\right]^2\left[\frac{-6(1+2x)}{(x^2+x+1)^2}\right] = \frac{-18(1+2x)^3}{(x^2+x+1)^4}$$

$$F_2(x) = 3U_0(x)U_1^2(x) + 3U_0^2(x)U_2(x) = 3\left[\frac{1+2x}{x^2+x+1}\right]\left[\frac{-6(1+2x)}{(x^2+x+1)^2}\right]^2 + 3\left[\frac{1+2x}{x^2+x+1}\right]^2\left[\frac{36(1+2x)}{(x^2+x+1)^3}\right]$$

$$= \frac{216(1+2x)^3}{(x^2+x+1)^5}$$

$$\vdots$$

$$U_0(x) = \frac{1+2x}{x^2+x+1}$$

$$U_1(x) = \frac{\partial^2}{\partial x^2}U_0(x) - 2F_0(x) = \frac{2(1+2x)(x^2+x-2)}{(x^2+x+1)^3} - 2\left[\frac{1+2x}{x^2+x+1}\right]^3 = \frac{-6(1+2x)}{(x^2+x+1)^2}$$

$$U_2(x) = \frac{1}{2}\left[\frac{\partial^2}{\partial x^2}U_1(x) - 2F_1(x)\right] = \frac{1}{2}\left[\frac{-36(4x^3+6x^2-1)}{(x^2+x+1)^4} + \frac{36(1+2x)^3}{(x^2+x+1)^4}\right] = \frac{36(1+2x)}{(x^2+x+1)^3}$$

$$U_3(x) = \frac{1}{3}\left[\frac{\partial^2}{\partial x^2}U_2(x) - 2F_2(x)\right] = \frac{1}{3}\left[\frac{216(1+2x)(5x^2+5x-1)}{(x^2+x+1)^5} - \frac{432(1+2x)^3}{(x^2+x+1)^5}\right] = \frac{-216(1+2x)}{(x^2+x+1)^4}$$

From (3.2), we have,

$$u(x,t) = U_0(x) + U_1(x)t + U_2(x)t^2 + U_3(x)t^3 + \cdots + U_n(x)t^n + \cdots$$

$$u(x,t)=\frac{1+2x}{x^2+x+1}-\frac{6(1+2x)}{(x^2+x+1)^2}t+\frac{36(1+2x)}{(x^2+x+1)^3}t^2-\frac{216(1+2x)}{(x^2+x+1)^4}t^3+\cdots$$

V. Conclusion

In this paper, we use ADM and DTM to solving the heat equation with power nonlinearity. We were able to find approximate solutions. The results of the test example show that the ADM results are equal to DTM. in addition to, the DTM is a very simple technique to solve the heat equation with power nonlinearity than the ADM.

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